Sturm-Liouville Series of Normal Functions in the Theory of Integral Equations.

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(Abstract.)

In a memoir read before the Society on May 13 last I proved a theorem* which may be stated as follows:—

Let $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$ *be a complete system of normal functions relating to a function* $K(s, t)$ *which is of positive type in the square* $a \leq s \leq b$, $a \leq t \leq b$; *and let* $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ *be the corresponding singular values. Then the series*

$$\frac{\psi_1(s)\psi_1(t)}{\lambda_1-\lambda}+\frac{\psi_2(s)\psi_2(t)}{\lambda_2-\lambda}+\ldots+\frac{\psi_n(s)\psi_n(t)}{\lambda_n-\lambda}+\ldots$$

converges absolutely and uniformly, and has for its sum function $K_\lambda(s, t)$, *the solving function of* $\kappa(s, t)$.

I now show that this leads to the following theorem relative to the expansion of an arbitrary function as a series of such normal functions:—

Let the arrangement of the normal functions $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$ *be such that the corresponding singular values are in non-decreasing order of magnitude; and let* $U(s)$, $L(s)$ *be respectively the upper and lower limits of indeterminacy of the series*

$$\psi_1(s)\int_a^b \psi_1(t)f(t)\,dt+\psi_2(s)\int_a^b \psi_2(t)f(t)\,dt+\ldots+\psi_n(s)\int_a^b \psi_n(t)f(t)\,dt+\ldots,$$

in which $f(s)$ *is any function having a Lebesgue integral in the interval* (a, b). *Then*

$$U(s) \geq \overline{\lim_{\lambda\to-\infty}}\, -\lambda\int_a^b K_\lambda(s, t)f(t)\,dt \geq \underline{\lim_{\lambda\to-\infty}}\, -\lambda\int_a^b K_\lambda(s, t)f(t)\,dt \geq L(s) \qquad (1)$$

It will be clear that the application of this theorem in any particular case requires the determination of an asymptotic formula for $K_\lambda(s, t)$ when λ is negative and numerically great. In the third section I proceed to obtain such a formula for the case in which $\kappa(s, t)$ is the Green's function of the equation

$$\frac{d^2u}{ds^2}+qu = 0$$

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* *Vide* 'Phil. Trans. Roy. Soc.,' Series A, vol. 209, p. 445.

for an assigned pair of boundary conditions at the end points of $(0, \pi)$. The only restriction placed upon the function of s denoted by q is that it should be continuous in $(0, \pi)$. By employing this formula I show that, *if* $f(s)$ *is any function which has a Lebesgue integral in* $(0, \pi)$

$$\overline{\omega(s)} \geq \overline{\lim_{\lambda \to -\infty}} - \lambda \int_0^{\pi} \mathrm{K}_{\lambda}(s, t) f(t)\, dt \geq \underline{\lim_{\lambda \to -\infty}} - \lambda \int_0^{\pi} \mathrm{K}_{\lambda}(s, t) f(t)\, dt \geq \underline{\omega(s)}, \qquad (2)$$

where $\overline{\omega(s)}$ *and* $\underline{\omega(s)}$ *are respectively the upper and lower bilateral limits of* $f(s)$ *at a point* s *of the open interval* $(0, \pi)$. These bilateral limits are defined as follows. Let $\chi_r(t)$ $(r = 1, 2)$ be any two functions which possess limited second derivatives in an interval $(0, \alpha)$ $(\alpha > 0)$, and are such that

$$\chi_r(0) = 0, \quad \chi'_r(0) = 1.$$

The upper limit of

$$f(s - \chi_1(t)) + f(s + \chi_2(t)), \qquad (3)$$

as t tends to zero, will, in general, assume different values as the functions $\chi_1(t)$, $\chi_2(t)$, are varied. The lower limit of these values is called the *upper bilateral limit* of $f(s)$ at the point s. Similarly, the upper limit of the values assumed by the lower limit of (3) is defined to be the *lower bilateral limit* of $f(s)$ at s.

In the case before us, the normal functions $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$ are the solutions of

$$\frac{d^2u}{ds^2} + (q + \mu) u = 0,$$

which, for $\mu = \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, satisfy the same pair of boundary conditions as $\kappa(s, t)$. The series

$$\psi_1(s) \int_0^{\pi} \psi_1(t) f(t)\, dt + \psi_2(s) \int_0^{\pi} \psi_2(t) f(t)\, dt + \ldots + \psi_n(s) \int_0^{\pi} \psi_n(t) f(t)\, dt + \ldots$$

is called a *canonical Sturm-Liouville series corresponding to* $f(s)$. It follows from (1) that we have

$$\mathrm{U}(s) \geq \overline{\lim_{\lambda \to -\infty}} - \lambda \int_0^{\pi} \mathrm{K}_{\lambda}(s, t) f(t)\, dt \geq \underline{\lim_{\lambda \to -\infty}} - \lambda \int_0^{\pi} \mathrm{K}_{\lambda}(s, t) f(t)\, dt > \mathrm{L}(s)$$

at each point of $(0, \pi)$. This, in conjunction with (2), enables us to state general theorems relative to the behaviour of canonical Sturm-Liouville series. Thus, defining the common value of the upper and lower bilateral limits at a point where they are equal to be the bilateral limit at that point, we have :—

At each point of the open interval $(0, \pi)$ *where* $\omega(s)$, *the bilateral limit of* $f(s)$, *exists*

$$\mathrm{U}(s) \geq \omega(s) \geq \mathrm{L}(s).$$

Again, taking the case in which $U(s) = L(s)$, we see that:—

The sum of a canonical Sturm-Liouville series corresponding to $f(s)$ at any point of the open interval $(0, \pi)$ where it converges lies between the upper and lower bilateral limits of $f(s)$ at the point.

In the later portions of this section these theorems are extended to the most general type of Sturm-Liouville series (§ 22, *et seq.*).

In the fourth section I consider the convergence of canonical Sturm-Liouville series, employing a method which is suggested by the proof of the theorem of II, § 4, and the asymptotic formula obtained at the commencement of the third section.

It is eventually shown that:—

At any point of the open interval $(0, \pi)$ each of the canonical Sturm-Liouville series corresponding to an assigned function which is integrable in $(0, \pi)$ in accordance with Lebesgue's definition has the same limits of indeterminacy.

In particular we have:—

If any one of the canonical Sturm-Liouville series corresponding to the function converges at a point of the open interval $(0, \pi)$, then all of them converge at this point, and all have the same sum.

It is also shown that:—

If any one of the canonical Sturm-Liouville series corresponding to the assigned function converges uniformly in a set of points which together with its limiting points is contained within $(0, \pi)$, then all of these series converge uniformly in the set.

From the two theorems just quoted sufficient conditions are obtained for the convergence and for the uniform convergence of these series. Finally, the corresponding results are obtained for the general Sturm-Liouville series.